Subthreshold behavior and avalanches in an exactly solvable Charge Density Wave system

David C. Kaspar
*Mathematics Department, University of California, Berkeley, CA 94720, USA**

Muhittin Mungan
Physics Department, Boğaziçi University, Bebek 34342 Istanbul, Turkey†

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We present a simplified charge density wave (CDW) model exhibiting a depinning transition with threshold force and configurations that are explicit. Due to the periodic boundary conditions imposed, the threshold configuration has a set of topological defects whose location and number depend on the realization of the random phases. These defects are responsible for the randomness of the threshold force, which, as the system size $L \to \infty$, approaches a Gaussian with variance scaling as L^{-1}. The approach to threshold proceeds via avalanches whose size dependence on the external driving force F is described by a record-breaking process. We find that the depinning transition in this model is a critical phenomenon, with the cumulative avalanche size (polarization) diverging near threshold as $(F_{\text{th}} - F)^{-2}$.

The motion of Charge Density Waves (CDW) belongs to a class of systems in which a deformable structure (elastic manifold) is driven by external forces through a medium of impurities (random medium). It has been argued that the depinning transition for CDWs is a *dynamic critical phenomenon*, a phase transition between the pinned and sliding states with the driving force as the control parameter [1]. Analytical results for the divergence of strains at the depinning transition [2], functional renormalization group calculations [3], and extensive numerical simulations of CDWs and similar systems in 1-3 dimensions [4, 5] support this claim. However, apart from the functional RG calculations, which are perturbative, and simulations, there is little evidence whether the depinning transition is indeed a critical phenomenon, particularly in $d = 1$. To address this question we introduce an exactly solvable 1d CDW model that exhibits the tell-tales of a critical phenomenon and allows us to understand the origin of criticality. The model is easily extended to higher dimensions but we will not pursue this in the present letter.

We begin with the CDW Hamiltonian:

$$\mathcal{H}(\{y_i\}) = \sum_i \frac{1}{2}(y_i - y_{i-1})^2 + V(y_i - \alpha_i) - F y_i, \quad (1)$$

where $V(x)$ is 1-periodic and α_i, the impurity phases, are *i.i.d.* uniform on $(-\frac{1}{2}, +\frac{1}{2})$. Following Narayan and Fisher [3], we choose

$$V(x) = (\lambda/2)(x - [\![x]\!])^2. \quad (2)$$

Here λ is the strength of the potential and $[\![x]\!]$ is the nearest integer to x. Write

$$m_i \equiv [\![y_i - \alpha_i]\!] \qquad \text{and} \qquad \tilde{y}_i \equiv y_i - \alpha_i - m_i, \quad (3)$$

for the *well number* and *well coordinate*, respectively, of y_i; the former records which parabolic well contains y_i and the latter the displacement $\in [-\frac{1}{2}, +\frac{1}{2})$ from the well's center. For large system size L with periodic boundary conditions, the static configurations are given as

$$\tilde{y}_i = \frac{\eta}{1 - \eta^2} \sum_{j \in \mathbb{Z}} \eta^{|i-j|}(\Delta\alpha_j + \Delta m_j) + F/\lambda, \quad (4)$$

where $0 \leq \eta \leq 1$ is defined via $2 + \lambda = \eta^{-1} + \eta$, and we will ignore the $O(\eta^L)$ error introduced by summing instead over a single period. The nonlinearity of the system is an admissibility condition for the m_i: (4) must give $\tilde{y}_i \in [-\frac{1}{2}, +\frac{1}{2})$. All static configurations are linearly stable unless a particle is at a cusp of V.

From (4) we see that if the force F is increased or decreased, the particles will translate *uniformly*. F can be increased until the first particle, say j, reaches a cusp, $\tilde{y}_j = 1/2$. Any further increase in F causes particle j to jump wells $m_j \to m_j + 1$, displacing

$$\tilde{y}_k \to \tilde{y}_k - \delta_{jk} + \frac{1 - \eta}{1 + \eta}\, \eta^{|j-k|}, \quad (5)$$

which may cause other particles to jump also. We assume that the dynamics of the particles are purely relaxational and that all changes in the force are sufficiently slow that we reach the static configurations.

Like the CDW models with sinusoidal potential [5], this model exhibits reversible and irreversible behavior. If we change the force by ΔF, causing one or more particles to jump wells before reaching another stable configuration, and then return F to its original value, it can be shown [6] that *none* of the particles jump back. On the other hand, the resulting configuration moves uniformly and without jumps, *i.e. reversibly*, in the interval of forces between F and $F + \Delta F$. This establishes an equivalence relation among configurations that share the same well numbers m_i but differ in their well-coordinates $\tilde{y}_i$ by an overall translation. We choose as representatives the *zero-force configurations* satisfying (4) with $F = 0$.

We simplify further by choosing η small: defining the rescaled well coordinate z_i as $\eta z_i = \tilde{y}_i - F/\lambda$, we obtain from (4)

$$z_i = \Delta\alpha_i + \Delta m_i + O(\eta) \quad (6)$$

We will refer to the CDW model, with $O(\eta^2)$ terms dropped ($O(\eta)$ in the rescaled coordinates), as the *toy model*. A set of exact results can be obtained for the toy

model; the proofs and additional results for the untruncated model will be given elsewhere [6].

It is readily shown that the process of increasing the external force and evolving the configurations to threshold can be described by the *zero-force algorithm* (ZFA), which iterates the following procedure:

(i) Record $z_{\max} = \max_i z_i$.

(ii) Find $j = \arg\max_i z_i$ [7], set $z_j \to z_j - 2$, $z_{j\pm1} \to z_{j\pm1} + 1$, and $m_j \to m_j + 1$.

(iii) If any $z_i > z_{\max}$ from (i), goto (ii).

The first execution of (ii) jumps the particle which would first reach the cusp if the force were increased (uniformly translating the particles) and subsequent executions resolve those particles which would be pulled across the cusp as a result of the first. This process always terminates, possibly with all sites having jumped exactly once, in which case the z_i are unchanged. This fixed-point is the threshold configuration, and the ZFA finds it in finitely many steps.

From the ZFA it is clear that the evolution to the depinning transition minimizes $\max_i z_i$ and indeed the threshold configuration is the solution of this optimization problem; this is true for the non-truncated CDW model as well. Equation (6) suggests that setting $\Delta m_i = -[\![\Delta\alpha_i]\!]$ would be favorable, but periodicity and the requirement that $m_i \in \mathbb{Z}$ usually prevent this. The threshold configuration depends on $S \equiv \sum_{i=0}^{L-1} [\![\Delta\alpha_i]\!]$ and is given as [6]:

$$\Delta m_i = -[\![\Delta\alpha_i]\!] + J_i - \delta_{ik_L} \qquad (7)$$

where J is an integer vector selected as follows:

(i) *Case* $S \geq 0$. $J_i = 1$ for the $S+1$ positions i which have smallest $\Delta\alpha_i - [\![\Delta\alpha_i]\!]$, $J_i = 0$ otherwise;

(ii) *Case* $S < 0$. $J_i = -1$ for the $|S| - 1$ positions i which have largest $\Delta\alpha_i - [\![\Delta\alpha_i]\!]$, $J_i = 0$ otherwise;

and k_L is an index defined by *(divisibility condition)*

$$k_L \equiv \sum_{i=0}^{L-1} i(-[\![\Delta\alpha_i]\!] + J_i) \pmod{L}. \qquad (8)$$

We will refer to the sites where $\Delta m_i + [\![\Delta\alpha_i]\!] \equiv \epsilon_i \neq 0$ as *defects* with *charge* ϵ_i.

Given the threshold configuration z_i^{th}, the corresponding threshold force is

$$F_{\mathrm{th}}\left(\{\alpha_i\}\right) = \lambda\left(1/2 - \eta z_{\max}^{\mathrm{th}}\right), \qquad (9)$$

where $z_{\max}^{\mathrm{th}} \equiv \max_i z_i^{\mathrm{th}}$. The term in parentheses on the RHS is the distance from the particle with maximum z_i to the cusp. Writing $\omega_i \equiv \Delta\alpha_i - [\![\Delta\alpha_i]\!]$ and using (6) and (7), we find

$$z_{\max}^{\mathrm{th}} = \begin{cases} \omega_{\sigma(S)} + 1 & \text{if } S \geq 0 \\ \omega_{\sigma(L-S)} & \text{if } S < 0 \end{cases} \qquad (10)$$

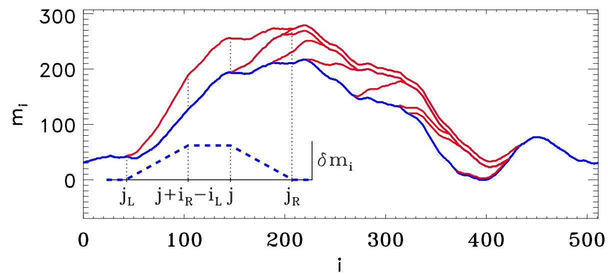

FIG. 1. Evolution of the m_i from negative (blue) to positive threshold configuration (top red curve) for $L = 512$. Approaching the threshold, increasingly large segments depin and advance. The inset shows the last configuration change $\delta m_i = m_i^+ - m_i$ and the corresponding labeling of the avalanche series. Note the trapezoidal shape of δm_i: the corners satisfy $\Delta\delta m_i = \pm 1$.

where σ is the permutation of the indices that orders ω: $\omega_{\sigma(0)} < \omega_{\sigma(1)} < \cdots < \omega_{\sigma(L-1)}$. The results for the threshold configuration and force characterize the manner of their dependence on the random phases α_i, via (a) the statistics of S and (b) the rank statistics of $\{\omega_i\}$ [8]. Routine arguments show $L^{-1/2}S$ converges in distribution as $L \to \infty$ to a normal random variable with mean 0 and variance $1/12$, so the typical number of topological defects scales as $L^{1/2}$. As a consequence the variance of the threshold force distribution averaged over the quenched disorder is found to scale as L^{-1} and thus fluctuations ΔF_{th} around the mean $\mathbb{E}F_{\mathrm{th}} = (1-\eta)^3/2\eta + O(L^{-3/2})$ scale as $L^{-1/2}$. This matches the scaling behavior of the sinusoidal CDW model [5], for which the expected behavior is $\Delta F_{\mathrm{th}} \sim L^{-1/\nu_T}$, with ν_T the finite-size scaling exponent from the scaling theory of Chayes *et al* [9]. Our model saturates their prediction $\nu_T \geq 2$ in 1d.

Equation (7) can be used to show that if the strains $s_i = m_{i+1} - m_i$ are rescaled as

$$s^{(L)}(t) \equiv (12/L)^{1/2} s_{\lfloor t/L \rfloor} \quad (0 \leq t \leq 1), \qquad (11)$$

the resulting càdlàg processes $s^{(L)}$ converge in distribution as $L \to \infty$ to a periodic Brownian motion with zero integral:

$$B(t) - \int_0^1 B(r)\,dr \quad (0 \leq t \leq 1), \qquad (12)$$

where $B(t)$ is a standard Brownian bridge. It follows that the typical maximum and minimum of (s_i) scale like $L^{1/2}$, matching our expectation for diverging strains at threshold as shown by Coppersmith [2].

We now consider the behavior of the system as it approaches threshold. As in [5] we start out with the configuration $\{z_i^-\}$ at negative threshold force, $F_{\mathrm{th}}^- < 0$, and increase the force until we reach the positive threshold configuration $\{z_i^+\}$. Figure 1 shows the corresponding evolution in terms of the well numbers $\{m_i\}$. The negative threshold configuration $\{z_i^-\}$ can be obtained from (7) by

switching $\alpha \to -\alpha$. For simplicity we consider the case $S = 1$ for which

$$z_i^- = \omega_i + \delta_{i,k^-}, \tag{13}$$
$$z_i^+ = \omega_i + \delta_{i,\sigma(0)} + \delta_{i,\sigma(1)} - \delta_{i,k^+}, \tag{14}$$

with $k^\pm$ determined by (8). Following the ZFA, the first site to jump will be k^- and $z_{\max} = \omega_{k^-} + 1$. As a result

$$z_{k^-} = \omega_{k^-} + 1 \to \omega_{k^-} - 1$$

and

$$z_{k^-\pm1} = \omega_{k^-\pm1} \to \omega_{k^-\pm1} + 1.$$

The neighboring sites $k^- \pm 1$ will be forced to jump if $\omega_{k^-\pm1} > \omega_{k^-}$. (Here and in the following addition and subtraction of indices are mod L.) It is readily shown that all those *consecutive* neighbors j to the left and right of k^- for which $\omega_j > \omega_{k^-}$ are forced to jump, giving rise to an avalanche with extents i_L and i_R, respectively, where $j_L = k^- - i_L - 1$ is the first site to the left of k^- for which $\omega_{j_L} < \omega_{k^-}$ and likewise for $j_R = k^- + i_L + 1$. Note that if a site j and both its neighbors jump, z_j is unchanged; in particular if $k^- \pm 1$ jump, then z_{k^-} is still the maximum and will initiate the next avalanche.

The above observations let us group several iterations in the ZFA which resolve all the avalanches initiated at a single site. To understand this process, observe that only the ranks $\sigma(j)$ (not the precise values ω_j) are required to determine the avalanche waves. From z_i we draw a *rank-defect diagram*, displaying the rank $\sigma(i)$ of $\omega(i)$ with over- or underlines to indicate positive or negative defects, respectively. Consider an example: let z^- have representation

$$\ldots\ 0\ \ 10\ \ 12\ \ 17\ \ \overline{15}\ \ 16\ \ 18\ \ 13\ \ 14\ \ 1\ \ \ldots$$

so that $\sigma(k^-) = 15$. The extent of the first avalanche will be $i_L = 1, i_R = 2$ and after the bracketed sites have jumped, the resulting configuration becomes

$$\ldots\ 0\ \ 10\ \ \overline{12}\ \ [\underline{17}\ \ \overline{15}\ \ 16\ \ \underline{18}]\ \ \overline{11}\ \ 13\ \ 1\ \ \ldots.$$

Next k^- and $k^- + 1$ will jump again, yielding

$$\ldots\ 0\ \ 10\ \ \overline{12}\ \ 17\ \ [15\ \ \underline{16}]\ \ 18\ \ \overline{11}\ \ 13\ \ 1\ \ \ldots,$$

No more avalanche waves are initiated at rank 15. Generalizing, given an initial site j, and initial extents (i_L, i_R), a new configuration will be reached after $\min(i_L, i_R) + 1$ avalanches, at the end of which the sites at j_L, j_R will have acquired a charge $+1$, while the sites k^- and $k^- + i_R - i_L$ will each acquire a charge -1. As a result, a total number of $(i_L + 1)(i_R + 1)$ jumps will have occurred. The next series of avalanches will be initiated by the rank 12 site with initial extent $(0, 2)$ and the configuration

$$\ldots\ 0\ \ \overline{10}\ \ [12\ \ 17\ \ \underline{15}]\ \ 16\ \ 18\ \ \overline{11}\ \ 13\ \ 1\ \ \ldots$$

is reached after $1 \times 3 = 3$ jumps. Next, the rank 11 and 10 sites trigger avalanche waves of extents $(2, 1)$ and $(0, 4)$, respectively, and we finally reach

$$\ldots\ \overline{0}\ \ 10\ \ 12\ \ 17\ \ 15\ \ \underline{16}\ \ 18\ \ 11\ \ 13\ \ \overline{1}\ \ \ldots.$$

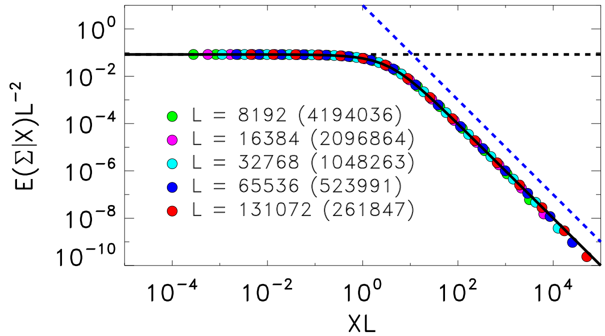

FIG. 2. Numerical results for the expectation of the cumulative jump size (polarization) plotted against the reduced force. Plot symbol colors refer to different system sizes as indicated in the legend. The numbers of realizations are indicated in parentheses. Scaling of the axes as indicated in the figure. The horizontal dashed line corresponds to (18), the slanted dashed line indicates a power law of exponent -2. The solid line is the finite size scaling function (20).

This is the threshold configuration as predicted by (14): the rank 1 particle would jump next, and the extent of this avalanche would be $(3, L-5)$ (periodic BC). The rank 16 site would have both neighbors jump and acquire a net positive defect, giving it a value above $z_{\max} = \omega_{\sigma(1)} + 1$, and it must also jump. Thus all particles jump, and the configuration is unchanged.

We now consider the statistics associated with this threshold-to-threshold evolution. Given a sequence of random variables $X_1, X_2, \ldots$, the site i is said to be a lower record, if $\min(X_1, X_2, \ldots, X_i) = X_i$. As indicated in the example, the sites initiating the avalanches are the lower records of the sequences

$$\begin{aligned} &\mathcal{J}_L = \omega_{k^-}, \omega_{k^--1}, \omega_{k^--2}, \ldots, \omega_{\sigma_L}, \\ \text{and} \quad &\mathcal{J}_R = \omega_{k^-}, \omega_{k^-+1}, \omega_{k^-+2}, \ldots, \omega_{\sigma_R}, \end{aligned} \tag{15}$$

where $\{\sigma_L, \sigma_R\} = \{\sigma(0), \sigma(1)\}$ are the termination sites [10]. The evolution from negative to positive threshold terminates when avalanches initiated at k^- and the left and right lower records reach σ_L and σ_R.

Observe that the active region where particle jumps occur is given by the segment containing k^- with boundaries σ_L and σ_R, and that the number of jumps leading to the τth configuration, whose left and right boundary have reached $j_L(\tau) \geq \sigma_L$ and $j_R(\tau) \leq \sigma_R$, is

$$\begin{aligned} \Sigma(\tau) &= \sum_i \left(m_i(\tau) - m_i^-\right) \\ &= \left(k^- - j_L(\tau)\right)\left(j_R(\tau) - k^-\right). \end{aligned} \tag{16}$$

In particular the total number of jumps in the threshold-to-threshold evolution is

$$\Sigma = \left(k^- - \sigma_L\right)\left(\sigma_R - k^-\right). \tag{17}$$

The random variables ω_i are exchangeable and statistics of records are well known in this case [11]. From this the

statistics of Σ and Σ_τ can be obtained. We defer details to [6] and focus here on expected values. By exchangeability σ_L and σ_R are selected uniformly from all (distinct) pairs of indices, and [6] k^- is uniform and independent of σ.

The expected total number of configurations $N_{\rm steps}$ in the threshold-to-threshold evolution is related to the expected number of records in the sequences (15), and one finds $\mathbb{E}N_{\rm steps} \sim 2\ln L - 4 + 2\gamma$, where γ is Euler's constant. The distribution of the total number of jumps Σ can be obtained from (17) and the statistics of k^-, σ_L and σ_R [6]. In particular,

$$\mathbb{E}\Sigma = L^2/12 + O(L). \tag{18}$$

Numerical results for the mean of the polarization (16) as a function of the reduced force $F_\tau - F^+_{\rm th} \sim z_{\rm max}(\tau) - z^{\rm th}_{\rm max}$, *cf.* (9), are shown in the scaling plot Fig. 2. The data collapse for different sizes L is perfect. To explain the scaling behavior and cross-over between the two regimes, we first require some notation.

Write $X = z_{\rm max} - z^{\rm th}_{\rm max} \in [0, 1)$ for the maximum height of the current configuration less that of the $(+)$-threshold configuration. Shift indices so that $k^- = 0$, and let $j_L(x)$ and $j_R(x)$ be the (noninclusive) left and right extents of the interval of sites that have jumped in the threshold-to-threshold evolution to achieve $X \le x$, chosen so that $-L < j_L \le 0 \le j_R < j_L + L$. Then the corresponding polarization is $\Sigma_x = -j_L(x)j_R(x)$, where $j_L(x)$ and $j_R(x)$ are the first indices in $\mathcal{J}_L$ and $\mathcal{J}_R$ of (15), respectively, for which $\omega_i \le \omega_{\sigma(1)} + x$.

For large L, approximate $\omega_{\sigma(1)} = -\frac{1}{2}$ and ω in $\mathcal{J}_L$ and $\mathcal{J}_R$ as *i.i.d.* uniform $(-\frac{1}{2}, +\frac{1}{2})$ variates [6], sharing their first elements. Their lengths ℓ_L and ℓ_R are random, independent of the sequence values, and uniform over all possible values. The random variables $|j_L|$ and j_R are geometric, truncated at ℓ_L and ℓ_R, and an easy calculation gives $\mathbb{E}[-j_L(x)j_R(x) \mid \ell_L, \ell_R]$:

$$\frac{1-x}{x^2}(1-(1-x)^{\ell_L})(1-(1-x)^{\ell_R}). \tag{19}$$

Setting $X = u/L$ and averaging over ℓ_L and ℓ_R, we obtain the finite-size scaling function $\Phi(u) \equiv \lim_{L\to\infty} L^{-2}\mathbb{E}[\Sigma_{u/L}]$:

$$\Phi(u) = \frac{6 - 4u + u^2 - 6e^{-u} - 2ue^{-u}}{u^4}. \tag{20}$$

This has no pole at $u = 0$; expanding the exponentials,

$$\Phi(u) = 1/12 - u/30 + u^2/120 - u^3/105 + O(u^4) \tag{21}$$

for $0 < u \ll 1$, and for $u \gg 1$ we have $\Phi(u) \sim u^{-2}$, with a crossover near $u = 1$. More intuitively, $j_R(u/L)$ is a geometric random variable with mean $L/u - 1$ truncated at ℓ_R, which has mean $L/3 + O(1)$. When $u \gg 1$ the truncation is largely irrelevant, since $j_R \ll \ell_R$ with high probability, and $j_R(u/L)$ scales as u^{-1}. As u nears 1 the truncation can no longer be ignored, and it dominates for $u \ll 1$, where $j_R(u/L)$ is insensitive to the exact value of u. In terms of finite-size scaling, for each of $-j_L(x)$ and $j_R(x)$ we have a correlation length $\xi \sim x^{-\nu}$ with finite-size scaling exponent $\nu = 1$ and finite-size effects dominating when $L/\xi \ll 1$. Even though j_L and j_R are dependent, the dependence is not so strong as to impact the scaling exponents of the product: they still combine to give scaling exponent $-\gamma + 1 = -2$ as in Fig. 2.

We have presented an exactly solvable 1d CDW toy model with a critical depinning transition, exposing the role played by the disorder and the boundary conditions. The approach to threshold is governed by a stochastic record-breaking process. The scaling exponents of the toy model differ from those obtained in terms of the $4 - \epsilon$ expansion of Narayan *et al.* [3, 5], in which $\nu = 2$ and $\gamma = 4$ are found in $d = 1$, but agree with those of the $d = 1$ CDW automaton model of Myers and Sethna [4]. These authors also note that for $d > 1$ the automaton model yields exponents in agreement with those obtained from the $4 - \epsilon$ expansion. We have also carried out numerical simulations for the full CDW model, (4), and find that the exponents ν and γ agree with those predicted by Narayan *et al.* Hence the full and toy CDW model exhibit different subthreshold critical behavior. We will report detailed results on the full model elsewhere. Lastly, it should be interesting to study the scaling behavior of the toy model presented here for $d > 1$.

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* kaspar@math.berkeley.edu
† mmungan@boun.edu.tr

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